Generalized Fractional Calculus of the H-Function (Applications of Complex Function Theory to Differential Equations)

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Generalized Fractional Calculus of the H-Function

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Abstract

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the H-function defined by the Mellin-Barnes integral

$$H_{p,q}^{m,n}(z)=\frac{1}{2\pi i}\int_{\mathfrak{L}}\mathcal{H}_{p,q}^{m,n}(s)z^{-s}ds,$$

where the function $\mathcal{H}_{p,q}^{m,n}(s)$ is a certain ratio of products of Gamma functions with the argument s and the contour $\mathfrak{L}$ is specially chosen. The considered generalized fractional integration and differentiation operators contain the Gauss hypergeometric function as a kernel and generalize classical fractional integrals and derivatives of Riemann-Liouville, Erdélyi-Kober type, etc. It is proved that the generalized fractional integrals and derivatives of H-functions are also H-functions but of greater order. In particular, the obtained results define more precisely and generalize known results.

1. Introduction

This paper deals with the H-function $H_{p,q}^{m,n}(z)$. For integers m, n, p, q such that $0 \leqq m \leqq q$, $0 \leqq n \leqq p$, for $a_i, b_j \in \mathbb{C}$ with $\mathbb{C}$ of the field of complex numbers and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$ $(i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)$ the H-function $H_{p,q}^{m,n}(z)$ is defined via a Mellin-Barnes type integral in the following way:

$$H_{p,q}^{m,n}(z)\equiv H_{p,q}^{m,n}\left[z\,\middle|\,\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right]\equiv H_{p,q}^{m,n}\left[z\,\middle|\,\begin{array}{c}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{array}\right]$$

$$=\frac{1}{2\pi i}\int_{\mathfrak{L}}\mathcal{H}_{p,q}^{m,n}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\,\middle|\,s\right]z^{-s}ds, \tag{1.1}$$

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where the contour $\mathfrak{L}$ is specially chosen and

$$\mathcal{H}_{p,q}^{m,n}(s) \equiv \mathcal{H}_{p,q}^{m,n}\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & s \\ (b_j,\beta_j)_{1,q} & \end{array}\right] = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, \quad (1.2)$$

in which an empty product, if it occurs, is taken to be one. Such a function was introduced by S. Pincherle in 1888 and its theory has been developed by Mellin [10], Dixon and Ferrar [2] (see [3, §1.19] in this connection). An interest to the H-function arose again in 1961 when Fox [4] has investigated such a function as a symmetrical Fourier kernel. Therefore this function is sometimes called as Fox's H-function. The theory of this function may be found in [1], [9, Chapter 1], [17, Chapter 2] and [11, 8.8.3].

Classical Riemann-Liouville fractional calculus of real order [17, §2.2] (see (2.1) - (2.6) below) was investigated in [12] - [14], [18] and [11]. The right-sided fractional integrals and derivatives of the H-function (1.1) were studied in [12] - [14] and the results were presented in [18, §2.7], where the case of left-sided fractional differentiation of the H-function was also considered. The left-sided fractional integration of the H-function was given in [11, 2.25.2]. Such results for the generalized fractional calculus operators with the Gauss hypergeometric function as a kernel (see (2.7) - (2.10) below), being introduced by the first author [15], were obtained in [16].

However, some of the results obtained in [12] - [14] (cited in [18]) and [16] can be taken to be more precisely. Moreover, these results were given provided that the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i > 0, \beta_j > 0$ $(i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)$ of the H-function satisfy certain conditions. These conditions were based on asymptotic behavior of $H_{p,q}^{m,n}(z)$ at zero and infinity. In [5] we extended such the known asymptotic results for the H-function to more wide class of parameters.

In [7], [8] we have applied the obtained asymptotic estimates in [5] to find the Riemann-Liouville fractional integrals and derivatives of any complex order of the H-function. In particular, we could make more precisely the known results from [12] - [14], [18] and [11].

The present paper is devoted to obtain such type results for the generalized fractional integration and differentiation operators of any complex order with the Gauss hypergeometric function as a kernel. In particular, we give more precisely some of the results from [16] and generalize the results obtained in [7], [8]. The paper is organized as follow. In Section 2 we present classical and generalized fractional calculus operators and some facts from the theory of Gauss hypergeometric function. Sections 3 and 4 contain the result from the theory of the H-function. The existence of $H_{p,q}^{m,n}(z)$ and its asymptotic behavior at zero and infinity is considered in Section 3 and certain reduction and differentiation properties in Section 4. Sections 5 and 6 deal with generalized fractional differentiation of the H-function (1.1). Sections 7 and 8 are devoted to the generalized fractional differentiation of the H-function. Another type of fractional integro-differentiation of the H-function is given in Section 9.

2. Classical and Generalized Fractional Calculus Operators

For $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0$, the Riemann-Liouville left- and right-sided fractional calculus operators are defined as follow [17, §2.3 and §2.4]:

$$\left(I_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > 0), \tag{2.1}$$

$$\left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x > 0), \tag{2.2}$$

and

$$\begin{aligned}\left(D_{0+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)]+1} \left(I_{0+}^{1-\alpha+[\mathrm{Re}(\alpha)]} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\mathrm{Re}(\alpha)])} \int_0^x \frac{f(t)}{(x-t)^{\alpha-[\mathrm{Re}(\alpha)]}}\, dt \quad (x > 0),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\left(D_{-}^{\alpha} f\right)(x) &= \left(-\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)]+1} \left(I_{-}^{1-\alpha+[\mathrm{Re}(\alpha)]} f\right)(x) \\ &= \left(-\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\mathrm{Re}(\alpha)])} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha-[\mathrm{Re}(\alpha)]}}\, dt \quad (x > 0),\end{aligned} \tag{2.4}$$

respectively, where the symbol $[\kappa]$ means the integral part of a real number κ, i.e. the largest integer not exceeding κ. In particular, for real $\alpha > 0$, the operators D_{0+}^{α} and D_{-}^{α} take more simple forms

$$\begin{aligned}\left(D_{0+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0+}^{1-\{\alpha\}} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}}\, dt \quad (x > 0),\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\left(D_{-}^{\alpha} f\right)(x) &= \left(-\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{-}^{1-\{\alpha\}} f\right)(x) \\ &= \left(-\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_x^\infty \frac{f(t)}{(t-x)^{\{\alpha\}}}\, dt \quad (x > 0),\end{aligned} \tag{2.6}$$

respectively, where $\{\kappa\}$ stands for the fractional part of κ, i.e. $\{\kappa\} = \kappa - [\kappa]$.

For $\alpha, \beta, \eta \in \mathbb{C}$ and $x > 0$ the generalized fractional calculus operators are defined by [15]

$$\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt \tag{2.7}$$

$$(\mathrm{Re}(\alpha) > 0);$$

$$\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+}^{\alpha+n,\beta-n,\eta-n} f\right)(x) \quad (\mathrm{Re}(\alpha) \leqq 0; n = [\mathrm{Re}(-\alpha)] + 1); \tag{2.8}$$

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t) dt \tag{2.9}$$

$$(\mathrm{Re}(\alpha) > 0);$$

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \left(-\frac{d}{dx}\right)^n \left(I_{-}^{\alpha+n,\beta-n,\eta} f\right)(x) \quad (\mathrm{Re}(\alpha) \leqq 0; n = [\mathrm{Re}(-\alpha)] + 1); \tag{2.10}$$

and

$$\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) \equiv \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta} f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f\right)(x) \quad (\mathrm{Re}(\alpha) > 0; n = [\mathrm{Re}(\alpha)] + 1); \tag{2.11}$$

$$\left(D_{-}^{\alpha,\beta,\eta} f\right)(x) \equiv \left(I_{-}^{-\alpha,-\beta,\alpha+\eta} f\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f\right)(x) \quad (\mathrm{Re}(\alpha) > 0; n = [\mathrm{Re}(\alpha)] + 1)). \tag{2.12}$$

Here ${}_2F_1(a, b; c; z)$ $(a, b, c, z \in \mathbb{C})$ is the Gauss hypergeometric function defined by the series

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{2.13}$$

with

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (k \in \mathbb{N}), \tag{2.14}$$

where $\Gamma(z)$ is the Gamma function [3, Chapter I] and $\mathbb{N}$ denotes the set of positive integers.

The series in (2.13) is convergent for $|z| < 1$ and for $|z| = 1$ with $\mathrm{Re}(c - a - b) > 0$, and can be analytically continued into $\{z \in \mathbb{C} : |\arg(1-z)| < \pi\}$ (see [3, Chapter II]).

Since

$${}_2F_1(0, b; c; z) = 1 \tag{2.15}$$

for $\beta = -\alpha$, the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide with the Riemann-Liouville operators (2.1) - (2.4) for $\mathrm{Re}(\alpha) > 0$:

$$\left(I_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0+}^{\alpha} f\right)(x), \qquad \left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x), \tag{2.16}$$

$$\left(D_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(D_{0+}^{\alpha} f\right)(x), \qquad \left(D_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(D_{-}^{\alpha} f\right)(x). \tag{2.17}$$

According to the relation [3, 2.8(4)]

$$ {}_2F_1(a,b;a;z) = (1-z)^{-b}, \tag{2.18}$$

when $\beta = 0$ the operators (2.7) and (2.9) coincide with the Erdélyi-Kober fractional integrals [17, §18.1]:

$$\left(I_{0+}^{\alpha,0,\eta}f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)dt \equiv \left(I_{\eta,\alpha}^{+}f\right)(x) \quad (\alpha,\eta\in\mathbb{C},\mathrm{Re}(\alpha)>0), \tag{2.19}$$

$$\left(I_{-}^{\alpha,0,\eta}f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\eta-\alpha}f(t)dt \equiv \left(K_{\eta,\alpha}^{-}f\right)(x) \quad (\alpha,\eta\in\mathbb{C},\mathrm{Re}(\alpha)>0). \tag{2.20}$$

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called "generalized" fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

$$D_{0+}^{\alpha,\beta,\eta} = \left(I_{0+}^{\alpha,\beta,\eta}\right)^{-1}, \qquad D_{-}^{\alpha,\beta,\eta} = \left(I_{-}^{\alpha,\beta,\eta}\right)^{-1}. \tag{2.21}$$

Fractional calculus operators (2.1), (2.3), (2.5), (2.7), (2.8), (2.11) and (2.2), (2.4), (2.6), (2.9), (2.10), (2.12) are called left-sided and right-sided, respectively [17, §2].

We give some other properties of ${}_2F_1(a,b;c;z)$ [3, 2.8(46), 2.9(2), 2.10(14)] which will be used in the following calculations:

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c\neq 0,-1,-2,\cdots;\mathrm{Re}(c-a-b)>0); \tag{2.22}$$

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z); \tag{2.23}$$

$$ {}_2F_1(a,b;a+b;z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(k!)^2}[2\psi(1+k)-\psi(a+k)+\psi(b+k)$$
$$-\log(1-z)](1-z)^k \quad (|\arg(z)|<\pi; a,b\neq 0,-1,-2,\cdots), \tag{2.24}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the Psi function [3, 1.7].

Formulas (2.22) - (2.24) mean the following asymptotic behavior of ${}_2F_1(a,b;c;z)$ at the point $z=1$.

Lemma 1. *For $a,b,c\in\mathbb{C}$ with $\mathrm{Re}(c)>0$ and $z\in\mathbb{C}$, there hold the following asymptotic relations near $z=1$:*

$$ {}_2F_1(a,b;c;z) = O(1) \quad (z\to 1-) \tag{2.25}$$

for $\mathrm{Re}(c-a-b)>0$;

$$ {}_2F_1(a,b;c;z) = O\left((1-z)^{c-a-b}\right) \quad (z\to 1-) \tag{2.26}$$

for $\mathrm{Re}(c-a-b)<0$; and

$$ {}_2F_1(a,b;c;z) = O\left(\log(1-z)\right) \quad (z\to 1-) \tag{2.27}$$

for $c-a-b=0,\ a,b\neq 0,-1,-2,\cdots$ *and* $|\arg(z)|<\pi$.

3. Existence and Asymptotic Behavior of the H-Function

We shall consider the H-function (1.1) provided that the poles

$$b_{jl}=\frac{-b_j-l}{\beta_j}\quad (j=1,\cdots,m;l\in\mathbb{N}_0) \tag{3.1}$$

of the Gamma functions $\Gamma(b_j+\beta_j s)$ and that

$$a_{ik}=\frac{1-a_i+k}{\alpha_i}\quad (i=1,\cdots,n;k\in\mathbb{N}_0) \tag{3.2}$$

of $\Gamma(1-a_i-\alpha_i s)$ do not coincide:

$$\alpha_i(b_j+l)\neq\beta_j(a_i-k-1)\quad (i=1,\cdots,n;j=1,\cdots,m;k,l\in\mathbb{N}_0), \tag{3.3}$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. $\mathfrak{L}$ in (1.1) is the infinite contour splitting poles b_{jl} in (3.1) to the left and poles a_{ik} in (3.2) to the right of $\mathfrak{L}$ and has one of the following forms:

(i) $\mathfrak{L}=\mathfrak{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty+i\varphi_1$ and terminating at the point $-\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$;

(ii) $\mathfrak{L}=\mathfrak{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty+i\varphi_1$ and terminating at the point $+\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$.

(iii) $\mathfrak{L}=\mathfrak{L}_{i\gamma\infty}$ is a contour starting at the point $\gamma-i\infty$ and terminating at the point $\gamma+i\infty$ with $\gamma\in\mathbb{R}=(-\infty,+\infty)$.

The properties of the H-function $H^{m,n}_{p,q}(z)$ depend on the numbers a^*,Δ,δ and μ which are expressed via p,q, a_i,α_i $(i=1,2,\cdots,p)$ and b_j,β_j $(j=1,2,\cdots,q)$ by the following relations:

$$a^*=\sum_{i=1}^{n}\alpha_i-\sum_{i=n+1}^{p}\alpha_i+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j, \tag{3.4}$$

$$\Delta=\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i, \tag{3.5}$$

$$\delta=\prod_{i=1}^{p}\alpha_i^{-\alpha_i}\prod_{j=1}^{q}\beta_j^{\beta_j}, \tag{3.6}$$

$$\mu=\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i+\frac{p-q}{2}. \tag{3.7}$$

Here an empty sum in (3.4), (3.5), (3.7) and an empty product in (3.6), if they occur, are taken to be zero and one, respectively.

The existence of the H-function is given by the following result [6].

Theorem A. *Let a^*, Δ, δ and μ be given by* (3.4) - (3.7). *Then the H-function $H^{m,n}_{p,q}(z)$ defined by* (1.1) *and* (1.2) *makes sense in the following cases:*

$$\mathfrak{L} = \mathfrak{L}_{-\infty}, \quad \Delta > 0, \quad z \neq 0; \tag{3.8}$$

$$\mathfrak{L} = \mathfrak{L}_{-\infty}, \quad \Delta = 0, \quad 0 < |z| < \delta; \tag{3.9}$$

$$\mathfrak{L} = \mathfrak{L}_{-\infty}, \quad \Delta = 0, \quad \mathrm{Re}(\mu) < -1, \quad |z| = \delta; \tag{3.10}$$

$$\mathfrak{L} = \mathfrak{L}_{+\infty}, \quad \Delta < 0, \quad z \neq 0; \tag{3.11}$$

$$\mathfrak{L} = \mathfrak{L}_{+\infty}, \quad \Delta = 0, \quad |z| > \delta; \tag{3.12}$$

$$\mathfrak{L} = \mathfrak{L}_{+\infty}, \quad \Delta = 0, \quad \mathrm{Re}(\mu) < -1, \quad |z| = \delta; \tag{3.13}$$

$$\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}, \quad a^* > 0, \quad |\arg z| < \frac{a^*\pi}{2}, \quad z \neq 0; \tag{3.14}$$

$$\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}, \quad a^* = 0, \quad \Delta\gamma + \mathrm{Re}(\mu) < -1, \quad \arg z = 0, \quad z \neq 0. \tag{3.15}$$

Remark 1. The results of Theorem A in the cases (3.10), (3.13) and (3.15) are more precisely than those in [11, §8.3.1].

The next statement being followed from the results in [5] characterizes the asymptotic behavior of the H-function at zero and infinity.

Theorem B. *Let a^* and Δ be given by* (3.4) *and* (3.5) *and let conditions in* (3.3) *be satisfied.*

(i) *If $\Delta \geqq 0$ or $\Delta < 0, a^* > 0$, then the H-function has either of the asymptotic estimates at zero*

$$H^{m,n}_{p,q}(z) = O\left(z^{\varrho^*}\right) \quad (|z| \to 0) \tag{3.16}$$

or

$$H^{m,n}_{p,q}(z) = O\left(z^{\varrho^*}[\log(z)]^{N^*}\right) \quad (|z| \to 0), \tag{3.17}$$

with the additional condition $|\arg(z)| < a^*\pi/2$ when $\Delta < 0, a^* > 0$. Here

$$\varrho^* = \min_{1 \leqq j \leqq m}\left[\frac{\mathrm{Re}(b_j)}{\beta_j}\right], \tag{3.18}$$

and N^* is the order of one of the point b_{jl} in (3.1) *to which some other poles of $\Gamma(b_j+\beta_j s)$ $(j = 1, \cdots, m)$ coincide.*

(ii) *If $\Delta \leqq 0$ or $\Delta > 0, a^* > 0$, then the H-function has either of the asymptotic estimates at infinity*

$$H^{m,n}_{p,q}(z) = O\left(z^{\varrho}\right) \quad (|z| \to \infty) \tag{3.19}$$

or

$$H_{p,q}^{m,n}(z) = O\left(z^{\varrho}[\log(z)]^{N}\right) \quad (|z| \to \infty), \tag{3.20}$$

with the additional condition $|\arg(z)| < a^*\pi/2$ *when* $\Delta > 0, a^* > 0$. *Here*

$$\varrho = \max_{1 \leqq i \leqq n}\left[\frac{\mathrm{Re}(a_i) - 1}{\alpha_i}\right], \tag{3.21}$$

and N *is the order of one of the point* a_{ik} *in* (3.2) *in which some other poles of* $\Gamma(1 - a_i - \alpha_i s)$ $(i = 1, \cdots, n)$ *coincide.*

4. Reduction and Differentiation Properties of the H-Function

In this and next sections we suppose that the conditions for the existence of the H-function given in Theorem A are satisfied.

The following two Lemmas which characterize symmetric and reduction properties of the H-function follow from the definition of the H-function in (1.1) - (1.2).

Lemma 2. *The H-function* (1.1) *is commutative in the set of pairs* $(a_1, \alpha_1), \cdots, (a_n, \alpha_n)$; *in* $(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$; *in* $(b_1, \beta_1), \cdots, (b_m, \beta_m)$ *and in* $(b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)$.

Lemma 3. *If one of* (a_i, α_i) $(i = 1, \cdots, n)$ *is equal to one of* (b_j, β_j) $(j = m + 1, \cdots, q)$ *(or one of* (a_i, α_i) $(i = n + 1, \cdots, p)$ *is equal to one of* (b_j, β_j) $(j = 1, \cdots, m)$*), then the H-function reduces to the lower order one, that is,* p, q *and* n *(or* m*) decrease by unity. Two such results have the forms*

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q-1}, (a_1, \alpha_1) \end{matrix}\right] = H_{p-1,q-1}^{m,n-1}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{2,p} \\ (b_j, \beta_j)_{1,q-1} \end{matrix}\right] \tag{4.1}$$

provided that $n \geqq 1$ *and* $q > m$, *and*

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p-1}, (b_1, \beta_1) \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] = H_{p-1,q-1}^{m-1,n}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p-1} \\ (b_j, \beta_j)_{2,q} \end{matrix}\right] \tag{4.2}$$

provided that $m \geqq 1$ *and* $p > n$.

The next differentiation formulae follow from the definition of the H-function given in (1.1) - (1.2) and from the functional equation for the Gamma function [3, §1.2(6)]

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}. \tag{4.3}$$

Lemma 4. *There hold the following differentiation formulae for $\omega, c \in \mathbb{C}, \sigma > 0$*

$$\left(\frac{d}{dz}\right)^k \left\{ z^\omega H^{m,n}_{p,q} \left[cz^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right\}$$

$$= z^{\omega-k} H^{m,n+1}_{p+1,q+1} \left[cz^\sigma \middle| \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega, \sigma) \end{matrix} \right], \tag{4.4}$$

$$\left(\frac{d}{dz}\right)^k \left\{ z^\omega H^{m,n}_{p,q} \left[cz^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right\}$$

$$= (-1)^k z^{\omega-k} H^{m+1,n}_{p+1,q+1} \left[cz^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma) \\ (k-\omega, \sigma), (b_j, \beta_j)_{1,q} \end{matrix} \right]. \tag{4.5}$$

5. Left-Sided Generalized Fractional Integration of the H-Function

In the following sections we treat the H-function (1.1) - (1.2) with $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}$ and under the assumptions $a^* > 0$ or $a^* = 0, \Delta\gamma + \operatorname{Re}(\mu) < -1$ for a^*, Δ, μ being given by (3.4), (3.5), (3.7).

Here we consider the left-sided generalized fractional integration $I^{\alpha,\beta,\eta}_{0+}$ defined by (2.7).

Theorem 1. *Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) \neq \operatorname{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \min_{1 \leqq j \leqq m} \left[\frac{\operatorname{Re}(b_j)}{\beta_j} \right] + \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta - \beta)] + 1 > 0, \tag{5.1}$$

$$\sigma\gamma < \operatorname{Re}(\omega) + \min[0, \operatorname{Re}(\eta - \beta)] + 1. \tag{5.2}$$

Then the generalized fractional integral $I^{\alpha,\beta,\eta}_{0+}$ of the H-function (1.1) exists and the following relation holds:

$$\left(I^{\alpha,\beta,\eta}_{0+} t^\omega H^{m,n}_{p,q} \left[t^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right)(x)$$

$$= x^{\omega-\beta} H^{m,n+2}_{p+2,q+2} \left[x^\sigma \middle| \begin{matrix} (-\omega, \sigma), (-\omega+\beta-\eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega+\beta, \sigma), (-\omega-\alpha-\eta, \sigma) \end{matrix} \right]. \tag{5.3}$$

Proof. By (2.7) we have

$$\left(I^{\alpha,\beta,\eta}_{0+} t^\omega H^{m,n}_{p,q} \left[t^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right)(x)$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\omega \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\right) H^{m,n}_{p,q} \left[t^\sigma \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] dt. \tag{5.4}$$

According to (2.25), (2.26), (3.16) and (3.17), the integrand in (5.4) for any $x > 0$ has the asymptotic estimate at zero

$$(x-t)^{\alpha-1}t^{\omega}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)H_{p,q}^{m,n}\left[t^{\sigma}\ \middle|\ \begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]$$
$$=O\left(t^{\omega+\sigma\varrho^*+\min[0,\mathrm{Re}(\eta-\beta)]}\right)\quad(t\to+0)$$

or

$$=O\left(t^{\omega+\sigma\varrho^*+\min[0,\mathrm{Re}(\eta-\beta)]}[\log(t)]^{N^*}\right)\quad(t\to+0).$$

Here ϱ^* is given by (3.18) and N^* is indicated in Theorem B(i). Therefore the condition (5.1) ensures the existence of the integral (5.4).

Applying (1.2), making the change of variable $t = x\tau$, changing the order of integration and taking into account the formula [11, §2.21.1.11]

$$\int_0^x t^{\alpha-1}(x-t)^{c-1}\ {}_2F_1\left(a,b;c;1-\frac{t}{x}\right)dt=\frac{\Gamma(c)\Gamma(\alpha)\Gamma(\alpha+c-a-b)}{\Gamma(\alpha+c-a)\Gamma(\alpha+c-b)}x^{\alpha+c-1}\tag{5.5}$$
$$(a,b,c,\alpha\in\mathbb{C},\mathrm{Re}(\alpha)>0,\mathrm{Re}(c)>0,\mathrm{Re}(\alpha+c-a-b)>0),$$

we obtain

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\omega}H_{p,q}^{m,n}\left[t^{\sigma}\ \middle|\ \begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x)$$
$$=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}t^{\omega}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)H_{p,q}^{m,n}\left[t^{\sigma}\ \middle|\ \begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]dt$$
$$=\frac{x^{-\alpha-\beta}}{2\pi i\Gamma(\alpha)}\int_{\mathfrak{L}}\mathcal{H}_{p,q}^{m,n}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\ \middle|\ s\right]ds\int_0^x(x-t)^{\alpha-1}t^{\omega-\sigma s}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)dt$$
$$=\frac{x^{\omega-\beta}}{2\pi i}\int_{\mathfrak{L}}\mathcal{H}_{p,q}^{m,n}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\ \middle|\ s\right]\frac{\Gamma(1+\omega-s\sigma)\Gamma(1+\omega-\beta+\eta-\sigma s)}{\Gamma(1+\omega-\beta-s\sigma)\Gamma(1+\omega+\alpha+\eta-\sigma s)}x^{-\sigma s}ds.\tag{5.6}$$

We note that since $\mathfrak{L}=\mathfrak{L}_{i\gamma\infty}$, $\mathrm{Re}(s)=\gamma$ and therefore the condition (5.2) ensures the existence of the Mellin-Barnes integral above. Hence in view of (1.2)

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\omega}H_{p,q}^{m,n}\left[t^{\sigma}\ \middle|\ \begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x)$$
$$=x^{\omega-\beta}H_{p+2,q+2}^{m,n+2}\left[x^{\sigma}\ \middle|\ \begin{matrix}(-\omega,\sigma),(-\omega+\beta-\eta,\sigma),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(-\omega+\beta,\sigma),(-\omega-\alpha-\eta,\sigma)\end{matrix}\right].\tag{5.7}$$

and in accordance with (1.1) we obtain (5.3) which completes the proof of Theorem 1.

Corollary 1.1. *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \min_{1 \leqq j \leqq m} \left[\frac{\mathrm{Re}(b_j)}{\beta_j} \right] + \mathrm{Re}(\omega) + 1 > 0, \tag{5.8}$$

$$\sigma\gamma < \mathrm{Re}(\omega) + 1. \tag{5.9}$$

Then the Riemann-Liouville fractional integral I_{0+}^{α} of the H-function (1.1) *exists and the following relation holds:*

$$\left(I_{0+}^{\alpha} t^{\omega} H_{p,q}^{m,n} \left[t^{\sigma} \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right)(x) = x^{\omega+\alpha} H_{p+1,q+1}^{m,n+1} \left[t^{\sigma} \,\middle|\, \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha, \sigma) \end{matrix} \right]. \tag{5.10}$$

Corollary 1.2. *Let $\alpha, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \min_{1 \leqq j \leqq m} \left[\frac{\mathrm{Re}(b_j)}{\beta_j} \right] + \mathrm{Re}(\omega) + \min[0, \mathrm{Re}(\eta)] + 1 > 0, \tag{5.11}$$

$$\sigma\gamma < \mathrm{Re}(\omega) + \min[0, \mathrm{Re}(\eta)] + 1. \tag{5.12}$$

Then the Erdélyi-Kober fractional integral $I_{\eta,\alpha}^{+}$ of the H-function (1.1) *exists and the following relation holds:*

$$\left(I_{\eta,\alpha}^{+} t^{\omega} H_{p,q}^{m,n} \left[t^{\sigma} \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \right)(x) = x^{\omega} H_{p+1,q+1}^{m,n+1} \left[x^{\sigma} \,\middle|\, \begin{matrix} (-\omega - \eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega - \alpha - \eta, \sigma) \end{matrix} \right]. \tag{5.13}$$

Remark 2. In the case $a^* > 0, \Delta \geqq 0$ the relation (5.3) was indicated in [16, (4.2)], but in the assumptions of the result the condition (5.2) of Theorem 1 should be added.

Remark 3. Corollary 1.1 coincides with Theorem 1 in [7]. For real $\alpha > 0$ and $a^* > 0$ the relation (5.10) was indicated in [11, 2.25.2.2], but the conditions of its validity have to be also corrected according to (5.8) and (5.9).

6. Right-Sided Generalized Fractional Integration of the H-Function

In this section we consider the right-sided generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ defined by (2.9).

Theorem 2. *Let $\alpha,\beta,\eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) \neq \mathrm{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1,\cdots,p; j = 1,\cdots,q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \max_{1\leqq i\leqq n}\left[\frac{\mathrm{Re}(a_i)-1}{\alpha_i}\right] + \mathrm{Re}(\omega) < \min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)], \tag{6.1}$$

$$\sigma\gamma > \mathrm{Re}(\omega) - \min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)]. \tag{6.2}$$

Then the generalized fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the H-function (1.1) exists and the following relation holds:

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x)$$

$$= x^{\omega-\beta} H_{p+2,q+2}^{m+2,n}\left[x^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p},(-\omega,\sigma),(-\omega+\alpha+\beta+\eta,\sigma)\\(-\omega+\beta,\sigma),(-\omega+\eta,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right]. \tag{6.3}$$

Proof. By (2.9) we have

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{\omega-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]dt. \tag{6.4}$$

Due to (2.25), (2.26), (3.19) and (3.20), the integrand in (6.4) for any $x > 0$ has the asymptotic at infinity

$$(t-x)^{\alpha-1}t^{\omega-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$= O\left(t^{\omega-\min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)]-1+\sigma\varrho}\right) \quad (t\to+\infty)$$

or

$$= O\left(t^{\omega-\min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)]-1+\sigma\varrho}[\log(t)]^{N}\right) \quad (t\to+\infty).$$

Here ϱ is given by (3.21) and N is indicated in Theorem B(ii). Therefore the condition (6.1) ensures the existence of the integral (6.4). Applying (1.2), making the change $t = 1/\tau$ and using (5.5), we obtain

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{\Gamma(\alpha)}\int_{1/x}^{\infty}\left(t-\frac{1}{x}\right)^{\alpha-1}t^{\omega-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{1}{tx}\right)H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]dt$$

$$= \frac{x^{1-\alpha}}{2\pi i\Gamma(\alpha)} \int_{\mathfrak{L}} \mathcal{H}^{m,n}_{p,q} \left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \middle| s \right] \tau^{\sigma s} ds$$

$$\cdot \int_0^x (x-\tau)^{\alpha-1} \tau^{\beta-\omega-1+\sigma s} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x}\right) d\tau$$

$$= \frac{x^{-\omega+\beta}}{2\pi i} \int_{\mathfrak{L}} \mathcal{H}^{m,n}_{p,q} \left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \middle| s \right] \frac{\Gamma(-\omega+\beta+\sigma s)\Gamma(-\omega+\eta+\sigma s)}{\Gamma(-\omega+\sigma s)\Gamma(-\omega+\alpha+\beta+\eta+\sigma s)} x^{\sigma s} ds. \quad (6.5)$$

Since $\mathfrak{L} = \mathfrak{L}_{i\gamma\infty}$, $\mathrm{Re}(s) = \gamma$ and therefore the condition (6.2) guarantees the existence of the Mellin-Barnes integral above. Replacing in (6.5) x by $1/x$, we obtain (6.3).

Corollary 2.1. *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \max_{1 \leqq i \leqq n} \left[\frac{\mathrm{Re}(a_i) - 1}{\alpha_i} \right] + \mathrm{Re}(\omega) + \mathrm{Re}(\alpha) < 0, \quad (6.6)$$

$$\sigma\gamma > \mathrm{Re}(\omega) + \mathrm{Re}(\alpha). \quad (6.7)$$

Then the Riemann-Liouville fractional integral I^α_- of the H-function (1.1) exists and the following relation holds:

$$\left(I^\alpha_- t^\omega H^{m,n}_{p,q} \left[t^\sigma \middle| \begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \right] \right)(x) = x^{\omega+\alpha} H^{m+1,n}_{p+1,q+1} \left[x^\sigma \middle| \begin{array}{c} (a_i,\alpha_i)_{1,p}, (-\omega,\sigma) \\ (-\omega-\alpha,\sigma), (b_j,\beta_j)_{1,q} \end{array} \right]. \quad (6.8)$$

Corollary 2.2. *Let $\alpha, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \max_{1 \leqq i \leqq n} \left[\frac{\mathrm{Re}(a_i) - 1}{\alpha_i} \right] + \mathrm{Re}(\omega) < \mathrm{Re}(\eta), \quad (6.9)$$

$$\sigma\gamma > \mathrm{Re}(\omega) - \mathrm{Re}(\eta). \quad (6.10)$$

Then the Erdélyi-Kober fractional integral $K^-_{\eta,\alpha}$ of the H-function (1.1) exists and the following relation holds:

$$\left(K^-_{\eta,\alpha} t^\omega H^{m,n}_{p,q} \left[t^\sigma \middle| \begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \right] \right)(x) = x^{\omega} H^{m,n+1}_{p+1,q+1} \left[x^\sigma \middle| \begin{array}{c} (a_i,\alpha_i)_{1,p}, (-\omega+\eta+\alpha,\sigma) \\ (-\omega+\eta,\sigma), (b_j,\beta_j)_{1,q} \end{array} \right]. \quad (6.11)$$

Remark 4. In the case $a^* > 0, \Delta \geqq 0$ the relation of the form (6.3) was indicated in [16, (4.3)]. But it includes a mistake and should be replaced by (6.3) with the conditions (6.1) and (6.2).

Remark 5. Corollary 2.1 coincides with Theorem 2 in [7]. For real $\alpha > 0$ and $a^* > 0$ the relation (6.8) was indicated in [18, (2.5)], but the conditions of its validity have to be also corrected in accordance with (6.6) and (6.7).

7. Left-Sided Generalized Fractional Differentiation of the H-Function

Now we treat the left-sided generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ given by (2.11).

Theorem 3. *Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\alpha+\beta+\eta) \neq 0$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \min_{1 \leqq j \leqq m} \left[\frac{\mathrm{Re}(b_j)}{\beta_j}\right] + \mathrm{Re}(\omega) + \min[0, \mathrm{Re}(\alpha+\beta+\eta)] + 1 > 0, \tag{7.1}$$

$$\sigma\gamma < \mathrm{Re}(\omega) + \min[0, \mathrm{Re}(\alpha+\beta+\eta)] + 1. \tag{7.2}$$

Then the generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the H-function (1.1) exists and the following relation holds:

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma} \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]\right)(x)$$

$$= x^{\omega+\beta} H_{p+2,q+2}^{m,n+2}\left[x^{\sigma} \middle| \begin{matrix} (-\omega, \sigma), (-\omega-\eta-\alpha-\beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega-\beta, \sigma), (-\omega-\eta, \sigma) \end{matrix}\right]. \tag{7.3}$$

Proof. Let $n = [\mathrm{Re}(\alpha)] + 1$. From (2.11) we have

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma} \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma} \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]\right)(x), \tag{7.4}$$

which exists according to Theorem 1 with α, β and η being replaced by $-\alpha+n, -\beta-n$ and $\alpha+\eta-n$, respectively. Then we find

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma} \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n x^{\omega+\beta+n} H_{p+2,q+2}^{m,n+2}\left[x^{\sigma} \middle| \begin{matrix} (-\omega, \sigma), (-\omega-\alpha-\beta-\eta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega-\beta-n, \sigma), (-\omega-\eta, \sigma) \end{matrix}\right]. \tag{7.5}$$

Taking into account the differentiation formula (4.4) we have

$$\left(D_{0+}^{\alpha,\beta,\eta}t^{\omega}H_{p,q}^{m,n}\left[t^{\sigma}\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]\right)(x)$$

$$= x^{\omega+\beta}H_{p+3,q+3}^{m,n+3}\left[x^{\sigma}\left|\begin{matrix}(-\omega-\beta-\eta,\sigma),(-\omega,\sigma),(-\omega-\alpha-\beta-\eta,\sigma),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(-\omega-\beta-\eta,\sigma),(-\omega-\eta,\sigma),(-\omega-\beta,\sigma)\end{matrix}\right.\right], \quad (7.6)$$

and Lemma 2 and the reduction relation (4.1) imply (7.3), which completes the proof of theorem.

Corollary 3.1. *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1,\cdots,p; j = 1,\cdots,q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy the conditions in (5.8) and (5.9). Then the Riemann-Liouville fractional derivative D_{0+}^{α} of the H-function (1.1) exists and the following relation holds:*

$$\left(D_{0+}^{\alpha}t^{\omega}H_{p,q}^{m,n}\left[t^{\sigma}\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]\right)(x) = x^{\omega-\alpha}H_{p+1,q+1}^{m,n+1}\left[x^{\sigma}\left|\begin{matrix}(-\omega,\sigma),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(-\omega+\alpha,\sigma)\end{matrix}\right.\right]. \quad (7.7)$$

Remark 6. For real $\alpha > 0$ and $a^* > 0$ the relation (7.3) was given in [18, (2.7.13)], but the conditions of its validity have to be corrected in accordance with (7.1) and (7.2).

Remark 7. Corollary 3.1 coincides with Theorem 3 in [7].

8. Right-Sided Generalized Fractional Differentiation of the *H*-Function

Here we deal with the right-sided generalized fractional derivative $D_{-}^{\alpha,\beta,\eta}$ given by (2.12).

Theorem 4. *Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\alpha+\beta+\eta) + [\mathrm{Re}(\alpha)] + 1 \neq 0$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1,\cdots,p; j = 1,\cdots,q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \max_{1\leqq i\leqq n}\left[\frac{\mathrm{Re}(a_i)-1}{\alpha_i}\right] + \mathrm{Re}(\omega) + \max[\mathrm{Re}(\beta) + [\mathrm{Re}(\alpha)] + 1, -\mathrm{Re}(\alpha+\eta)] < 0, \quad (8.1)$$

$$\sigma\gamma > \mathrm{Re}(\omega) + \max[\mathrm{Re}(\beta) + [\mathrm{Re}(\alpha)] + 1, -\mathrm{Re}(\alpha+\eta)]. \quad (8.2)$$

Then the generalized fractional derivative $D_{-}^{\alpha,\beta,\eta}$ of the H-function (1.1) exists and the following relation holds:

$$\left(D_{-}^{\alpha,\beta,\eta}t^{\omega}H_{p,q}^{m,n}\left[t^{\sigma}\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]\right)(x)$$

$$= (-1)^{[\mathrm{Re}(\alpha)]+1}x^{\omega+\beta}H_{p+2,q+2}^{m+2,n}\left[x^{\sigma}\left|\begin{matrix}(a_i,\alpha_i)_{1,p},(-\omega,\sigma),(-\omega-\beta+\eta,\sigma)\\(-\omega-\beta,\sigma),(-\omega+\alpha+\eta,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]. \quad (8.3)$$

Proof. Let $n = [\mathrm{Re}(\alpha)] + 1$. Owing to (2.12) we have

$$\left(D_{-}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x), \tag{8.4}$$

which exists according to Theorem 2 with α, β and η being replaced by $-\alpha+n, -\beta-n$ and $\alpha+\eta$, respectively. Then applying the differentiation formula (4.5), similarly to (7.5), (7.6), we find in view of the reduction formula (4.2) that

$$\left(D_{-}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^n x^{\omega+\beta+n} H_{p+2,q+2}^{m+2,n}\left[x^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p},(-\omega,\sigma),(-\omega-\beta+\eta,\sigma)\\(-\omega-\beta-n,\sigma),(-\omega+\alpha+\eta,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$= (-1)^n x^{\omega+\beta} H_{p+3,q+3}^{m+3,n}\left[x^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p},(-\omega,\sigma),(-\omega-\beta+\eta,\sigma),(-\omega-\beta-n,\sigma)\\(-\omega-\beta,\sigma),(-\omega-\beta-n,\sigma),(-\omega+\alpha+\eta,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right],$$

which implies the formula (8.3).

Corollary 4.1. *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, and let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \max_{1 \leqq i \leqq n}\left[\frac{\mathrm{Re}(a_i) - 1}{\alpha_i}\right\} + \mathrm{Re}(\omega) - \{\mathrm{Re}(\alpha)\} + 1 < 0, \tag{8.5}$$

$$\sigma\gamma + \mathrm{Re}(\omega) - \{\mathrm{Re}(\alpha)\} + 1 > 0. \tag{8.6}$$

Then the Riemann-Liouville fractional derivative D_{-}^{α} of the H-function (1.1) exists and there holds the relation:

$$\left(D_{-}^{\alpha} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x) \tag{8.7}$$

$$= (-1)^{[\mathrm{Re}(\alpha)]+1} x^{\omega-\alpha} H_{p+1,q+1}^{m+1,n}\left[x^{\sigma}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p},(-\omega,\sigma)\\(-\omega+\alpha,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right]. \tag{8.8}$$

Remark 8. The relation of the form (8.7) with real $\alpha > 0$ and $a^* > 0$ was proved in [13, formula (14a)] (see also [12], [14, (2.2)] and [18, (2.7.9)]). But such a formula contains mistakes and should be replaced by (8.7) with the condition (8.5) and (8.6).

Remark 9. When $\alpha = k \in \mathbb{N}$, the relations (7.7) and (8.7) coincide with (4.4) and (4.5), respectively.

9. Generalized Fractional Integro-Differentiation of the H-Function

Here we investigate the generalized fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$ given by (2.8) and (2.10). The following statements are proved similarly to Theorems 3 and 4 by using the relations (2.8) and (2.10), Theorems 1 and 2, and the properties of the H-function in Sections 3 and 4.

Theorem 5. *Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \leqq 0, \mathrm{Re}(\beta) \neq \mathrm{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \min_{1 \leqq j \leqq m} \left[\frac{\mathrm{Re}(b_j)}{\beta_j}\right] + \mathrm{Re}(\omega) + \min[0, \mathrm{Re}(\eta - \beta)] + 1 > 0, \tag{9.1}$$

$$\sigma\gamma < \mathrm{Re}(\omega) + \min[0, \mathrm{Re}(\eta - \beta)] + 1. \tag{9.2}$$

Then the generalized fractional integro-differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the H-function (1.1) *exists and there holds the relation*

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma} \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]\right)(x)$$

$$= x^{\omega-\beta} H_{p+2,q+2}^{m,n+2}\left[x^{\sigma} \middle| \begin{matrix} (-\omega, \sigma), (-\omega - \eta + \beta, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\omega + \beta, \sigma), (-\omega - \alpha - \eta, \sigma) \end{matrix}\right]. \tag{9.3}$$

Theorem 6. *Let $\alpha, \beta, \eta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \leqq 0, \mathrm{Re}(\beta) + [\mathrm{Re}(\alpha)] - 1 \neq \mathrm{Re}(\eta)$. Let the constants $a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j > 0$ $(i = 1, \cdots, p; j = 1, \cdots, q)$ and $\omega \in \mathbb{C}, \sigma > 0$ satisfy*

$$\sigma \max_{1 \leqq i \leqq n} \left[\frac{\mathrm{Re}(a_i) - 1}{\alpha_i}\right] + \mathrm{Re}(\omega) < \min[\mathrm{Re}(\beta) - [\mathrm{Re}(-\alpha)] - 1, \mathrm{Re}(\eta)], \tag{9.4}$$

$$\sigma\gamma > \mathrm{Re}(\omega) - \min[\mathrm{Re}(\beta) - [\mathrm{Re}(-\alpha)] - 1, \mathrm{Re}(\eta)]. \tag{9.5}$$

Then the generalized fractional integro-differentiation $I_{-}^{\alpha,\beta,\eta}$ of the H-function (1.1) *exists and there holds the relation*

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\omega} H_{p,q}^{m,n}\left[t^{\sigma} \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right]\right)(x)$$

$$= x^{\omega-\beta} H_{p+2,q+2}^{m+2,n}\left[x^{\sigma} \middle| \begin{matrix} (a_i, \alpha_i)_{1,p}, (-\omega, \sigma), (-\omega + \alpha + \beta + \eta, \sigma) \\ (-\omega + \beta, \sigma), (-\omega + \eta, \sigma), (b_j, \beta_j)_{1,q} \end{matrix}\right]. \tag{9.6}$$

Remark 10. The relation (9.3) with $a^* > 0, \Delta \geqq 0$ was indicated in [16, (4.2)], but conditions of its validity have to be corrected in accordance with (9.1) and (9.2).

Remark 11. The relations (9.3) and (9.6) for the fractional integro-differentiation operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$, defined in (2.8) and (2.10) for $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) \leqq 0$ coincide with that (5.3) and (6.3) for the fractional integration operators $I_{0+}^{\alpha,\beta,\eta}$ and $I_{-}^{\alpha,\beta,\eta}$, defined in (2.7) and (2.9) for $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0$. Though the conditions for validity of (5.3) and (9.3) in Theorems 1 and 5 have the same form, that of (6.3) and (9.6) presented in Theorems 2 and 4 are slightly different.

In conclusion we note that, as it was mentioned in Remarks 2, 4 and 10, the relations (5.3), (6.3) and (9.3) for generalized calculus operator $I_{0+}^{\alpha,\beta,\eta}$ were already known in the case $a^* > 0, \Delta \geqq 0$. Further, Remarks 3,5,6 and 8 indicate that the relations (5.10) and (6.8) for the Riemann-Liouville fractional integrals $I_{0+}^{\alpha}, I_{-}^{\alpha}$ and (7.3) and (8.7) for the fractional derivative D_{0+}^{α} in the case real $\alpha > 0$ and $a^* > 0$ were established. However, the H-function's asymptotic estimates (3.16), (3.17) at zero and (3.19), (3.20) at infinity allow us to prove such results under more general assumptions $a^* > 0$ and $a^* = 0, \Delta\gamma + \mathrm{Re}(\mu) < -1$.

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